Jumping Frogs - Challenge #2

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1 Abstract

Here is presented a procedure for solving an arbitrary sequence of lily pads (here called a "pond") in which each lily pad has a single frog or is water-logged (which is the same as being empty). The solution is reached here by appending more single-frog lily pads to the right of the pond. By symmetry the frogs could be added to the left too.

Thus the problem is solvable for all water-logged configurations and an upper bound is implicitly set by the given procedures.

2 Notation

Frogs will be presented as lists of integers (within brackets where clarity is needed). For example, (207[15]0) is a 2-frog stack, an empty lily pad, a 7-frog stack, a 15-frog stack, and another empty lily pad.

For generalization, variables will be used as exponents to represent consecutive stacks of the same size and in stack sizes. For example, in $(33[n-2]0^k)$, the term $[n-2]$ would be read as a stack of $n-2$ frogs and the term 0^k would be read as k consecutive empty lily pads.

Water-logged lily pads will be represented with X's and have the same behavior as an empty lily pad. The $*$ character will represent a lily pad of unknown value.

Jumps are shown as follows:
for $a_i = k$ and $a_{i+k} > 0$

$$(a_1...a_{i-1}ka_{i+1}...a_{i+k}...a_n) \rightarrow (a_1...0...[a_{i+k}+k]...a_n)$$

and by symmetry:

$$(a_1...a_{i-k}...a_{i-1}ka_{i+1}...a_n) \rightarrow (a_1...[a_{i-k}+k]...0...a_n)$$

That is to say, a stack of k frogs can jump k spaces to the right or left.
Some examples:

$$(1111) \to (0211) \to (0202) \to (0400)$$

$$(1^n n) = (11^{n-1}n) \to ([n+1]1^{n-1}0) = ([n+1]11^{n-2}0) \to ([n+2]01^{n-2}0)$$

$$(1**1111) \to (1**1120) \to (1**3100) \to (4**0100) \to (0**0500)$$

3 Some Lemmas

Lemma 1.1: $(1^n) \to (n0^{n-1})$ for $n \geq 1$
Convert n consecutive single frogs into a stack of n frogs followed by $n-1$ consecutive empty lily pads.

$$(1^n) \to (1...1021...1) \to (1...13001...1) \to (1...100041...1) \to ...$$

$$... \to (1...10^{k-1}k1...1) \to ... \to (n0^{n-1})$$

Note: 0^{n-1} appears in the derivation. Thus this procedure requires $n \geq 1$.

Lemma 1.2: $(1^n) \to (0^{n-1}n)$ for $n \geq 1$
Apply symmetry to the result of Lemma 1.1a

Lemma 2.1: $(1 *^n 1^{n+5}) \to (0 *^n 0^{n+2}11[n+4])$ for $n \geq 2$
-Procedure to rescue a single frog n lily pads away.
-Appends $n+5$ single-frog lily pads to the right:

$$(1 *^n 1^{n+5}) = (1 *^n 111^{n-1}1111) \xrightarrow{1.2} (1 *^n 110^{n-2}[n-1]1111) \to$$

$$\to (1 *^n 110^{n-2}n0111) \to (1 *^n [n+1]10^{n-2}00111) = (1 *^n [n+1]10^n 111) \to$$

$$\to ([n+2] *^n 010^n 111) \to (0 *^n 0[n+3]0^n 111) \to (0 *^n 0^{n+2}11[n+4])$$

Note: 0^{n-2} appears in the derivation. Thus this procedure requires $n \geq 2$.

Lemma 2.2: $(1 *^n 1^{n+5}) \to (0 *^n 0^{n+4}[n+6])$ for $n \geq 2$
Follows immediately from Lemma 2.1:

$$(1 *^n 1^{n+5}) \xrightarrow{2.1} (0 *^n 0^{n+2}11[n+4]) \to (0 *^n 0^{n+2}20[n+4]) \to$$

$$\to (0 *^n 0^{n+2}00[n+6]) = (0 *^n 0^{n+4}[n+6])$$

Lemma 3.1: $(1 *^n m1^{m+n+6}) \to (0 *^n 0^{m+n+6}[2m+n+7])$ for $m \geq 16$, $n \geq 1$
-Procedure to rescue a single frog n lily pads away when there is a stack of frogs

on the right end of the pond.
-Appends $m+n+6$ single-frog lily pads to the right:

$$(1 *^n m1^{m+n+6}) =$$

$$= (1 *^n m1^{n+6}1^m) \xrightarrow{2.1}$$

$$\xrightarrow{2.1} (0 *^n m0^{n+3}11[n+5]1^m) \rightarrow$$

$$\rightarrow (0 *^n m0^{n+3}10[n+6]1^m) \rightarrow$$

$$\rightarrow (0 *^n [m+n+6]0^{n+3}1001^m) =$$

$$= (0 *^n [m+n+6]0^{n+3}1001^{m-1}1) \rightarrow$$

$$\rightarrow (0 *^n 00^{n+3}1001^{m-1}[m+n+7]) =$$

$$= (0 *^n 0^{n+4}1001^7 1^{m-8}[m+n+7]) \xrightarrow{2.2}$$

$$\xrightarrow{2.2} (0 *^n 0^{n+4}0000^6 81^{m-8}[m+n+7]) =$$

$$= (0 *^n 0^{n+13}81^{m-15}1^7[m+n+7]) \xrightarrow{1.1}$$

$$\xrightarrow{1.1} (0 *^n 0^{n+13}81^{m-15}70^6[m+n+7]) \xrightarrow{1.2}$$

$$\xrightarrow{1.2} (0 *^n 0^{n+13}80^{m-16}[m-15]70^6[m+n+7]) \rightarrow$$

$$\rightarrow (0 *^n 0^{n+13}[m-7]0^{m-16}070^6[m+n+7]) \rightarrow$$

$$\rightarrow (0 *^n 0^{n+13}[m-7]0^{m-16}000^6[m+n+14]) =$$

$$= (0 *^n 0^{n+13}[m-7]0^{m-8}[m+n+14]) \rightarrow$$

$$\rightarrow (0 *^n 0^{n+13}00^{m-8}[m+n+14+(m-7)]) =$$

$$= (0 *^n 0^{m+n+6}[2m+n+7])$$

Note: 0^{m-16} appears in the derivation. Thus this procedure requires $m \geq 16$.

4 Water-logged Rescue Procedure

Consider an arbitrary sequence of water-logged lily pads and single-frog lily pads, say $p = \{X, 1\}^*$. We write p in the form

$$p = (X^{a_k}1X^{a_{k-1}}1X^{a_{k-2}}1...1X^{a_2}1X^{a_1}1X^{a_0})$$

where X is a water-logged lily pad and $a_i \geq 0$ for all i.

Frog #1 Rescue
Case $a_0 = 0$: $(X^{a_k}1X^{a_{k-1}}1...1X^{a_1}1)$
No frogs are appended.

Case $a_0 = 1$: $(X^{a_k}1X^{a_{k-1}}1...1X^{a_1}1X)$

12 frogs are appended:

$$(X^{a_k}1X^{a_{k-1}}1...1X^{a_1}1X1^{12}) \to (X^{a_k}1X^{a_{k-1}}1...1X^{a_1}0X0^{11}[13])$$

Procedure is as follows:

$$(1X1^{12}) = (1X111111111111) \to (1X111112011111) \to$$

$$\to (1X111103011111) \to (1X114100011111) \to (5X110100011111) \to$$

$$\to (0X110600011111) \to (0X020600011111) \to (0X000800011111) \to$$

$$(0X000000011119) \xrightarrow{1.1} (0X000000040009) \to (0X0^{11}[13])$$

Case $a_0 \geq 2$: $(X^{a_k}1X^{a_{k-1}}1...1X^{a_1}1X^{a_0})$
$a_0 + 5$ frogs are appended.

$$(X^{a_k}1X^{a_{k-1}}1...1X^{a_1}1X^{a_0}1^{a_0+5}) \to (X^{a_k}1X^{a_{k-1}}1...1X^{a_1}0X^{a_0}0^{a_0+4}[a_0+6])$$

Frog #2-k Rescue

Regarding the first frog, all cases leave the pond in the following state:

$$(X^{a_k}1X^{a_{k-1}}1...1X^{a_2}1*^n m)$$

where $m \geq 1$, $n \geq 0$, and $*^n = X^{a_1}0X^{a_0}(0...0?)$.

But to apply Lemma 3.1 we need $m \geq 16$ and $n \geq 1$. These constraints can be met by repeatedly appending m frogs and jumping the m stack to the right. Procedure is as follows:

$$(*^n m1^m) = (*^n m1^{m-1}1) \to (*^n 01^{m-1}[m+1]) \xrightarrow{1.1} (*^n 0[m-1]0^{m-2}[m+1]) \to$$

$$\to (*^n 000^{m-2}[m+1+(m-1)] = (*^n 0^m[2m]) = (*^{n+m}[2m]) = (*^{n'}m')$$

where $m' = 2m$ and $n' = n + m$

Worst case is 5 applications of the former procedure when $m = 1$.

Apply Lemma 3.1 as follows:

$$(X^{a_k}1X^{a_{k-1}}1...1X^{a_2}1*^n m1^{m+n+6}) \xrightarrow{3.1}$$

$$\xrightarrow{3.1} (X^{a_k}1X^{a_{k-1}}1...1X^{a_2}0*^n 0^{m+n+6}[2m+n+7])$$

From here we repeatedly apply lemma 3.1; rescuing another frog from the water-logged area each time.

For $(X^{a_k}1X^{a_{k-1}}1...1X^{a_i}0*^{n'} 0^{m'+n'+6}[2m'+n'+7])$ choose $*^n = X^{a_i}0*^{n'} 0^{m'+n'+6}$ and $m = [2m'+n'+7]$. Apply the lemma as follows:

$$X^{a_k}1X^{a_{k-1}}1...1*^n m) \xrightarrow{3.1} (X^{a_k}1X^{a_{k-1}}1...1X^{a_{i+1}}0*^n 0^{m+n+6}[2m+n+7])$$

After applying the lemma $k-1$ times in total, the pond will be of the form $X^{a_k}*^n m$ where the $*^n$ is composed of water-logged and empty lily pads. m is the only frog stack and the configuration has been solved.

5 On Bounds

The recurrence relation between m and n is

$$\begin{bmatrix} m \\ n \end{bmatrix} \xrightarrow{\text{rescue}\#i} \begin{bmatrix} 2m + n + 7 \\ m + 2n + 7 + a_{i-1} \end{bmatrix}$$

Upon each "rescue", $m + n + 6$ frogs are appended. Thus the upper bound for the minimum number of frogs appended is $O(3^{\#\text{frogs}})$.

This should be **easily** beatable for any water-logged configuration. But it serves to show that all configurations are solvable.

The tight bound is probably $\Theta((1 + \epsilon)^{\#\text{frogs}})$