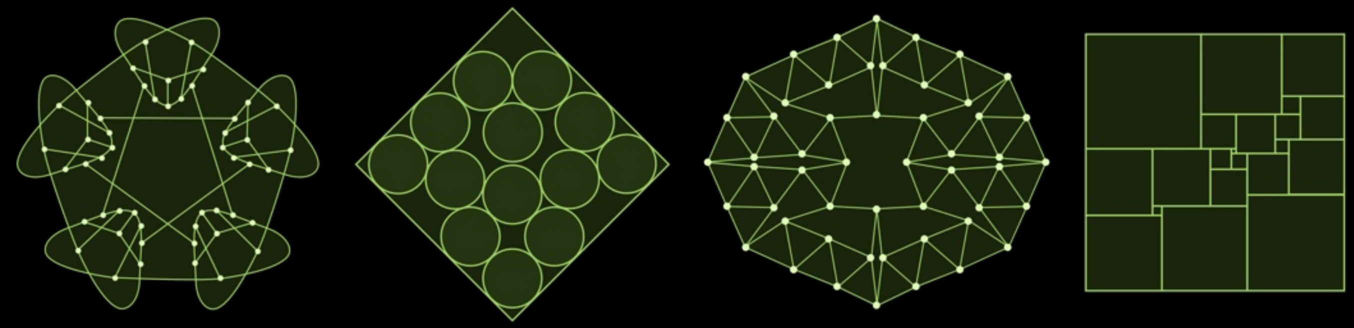


Unsolved K-12

A conference for mathematicians and educators

Nov 15-17 2013

Banff International Research Station

In November 2013, mathematicians and educators gathered to select one unsolved problem for each grade K-12. Here are the 13 winners...

The objective of the 2013 K-12 conference was to select one unsolved problem for each grade K-12. Some of these may be inspiring for this conference. In the following 4 pages I'll present the 13 winners - then a selection of other unsolved problems.

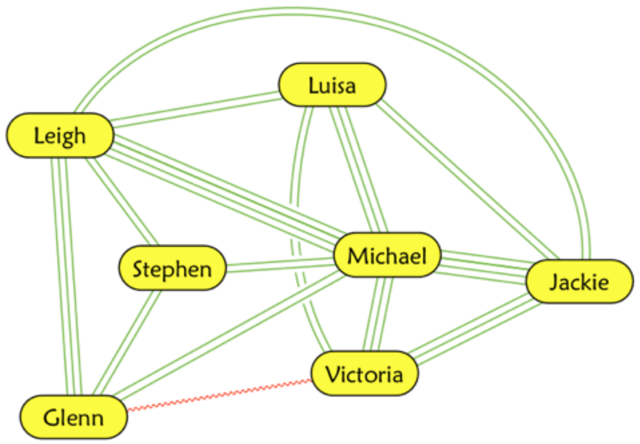
Kindergarten

Maximum Induced Planar Subgraph Problem

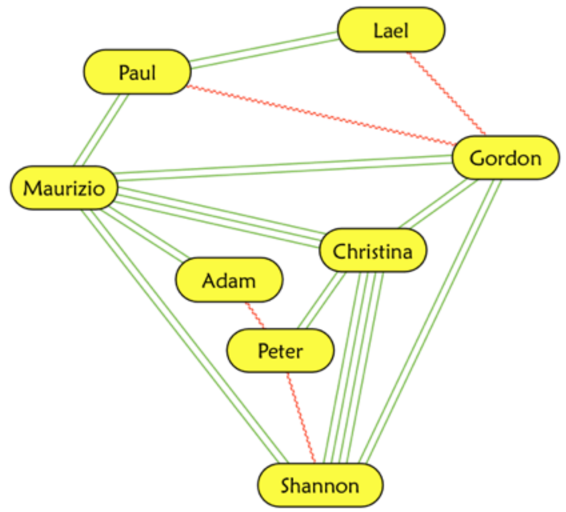
M. Krishnamoorthy & N. Deo, 1977

M. K.

Children write some of their names on the whiteboard. Children are "Friends" if they share at least two letters. Connect them by green lines. Children are "Enemies" if they share no letters. Connect these children with a jagged red line. Laugh about what a ridiculous way this is for you to make friends and enemies. Tell the children that they win by getting the most people on the board with no overlapping connections.



Failure. Victoria and Luisa are connected by an overlapped line.



Success. This is a planar graph, but is it as large as possible?

Unsolved Problem: Find the most efficient algorithm to solve these problems. Currently the record holder is the paper on the next page.

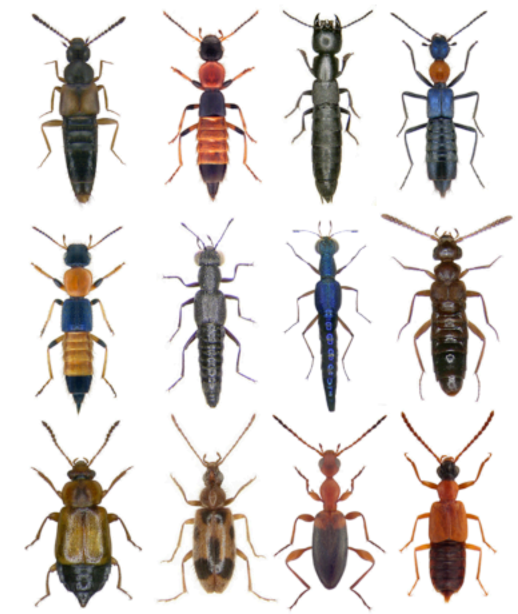
Grade 1

Optimal Neural Network

ImageNet, 2013

ImageNet hosts annual competitions to solve practical unsolved problems about pattern recognition. Neural networks are trained on one set of categorized ImageNet photographs and are then tested on a new set of images to see if they can guess the correct category. See image-net.org for details. Kindergarten students should also be engaged in a rich discussion of sorting. These photographs from retired beetle collector Udo Schmidt are an ideal data set.

Collect beetles from the Staphylinidae family.



There are 2 impostors from the Anthicidae family.

Winning Unsolved Problems

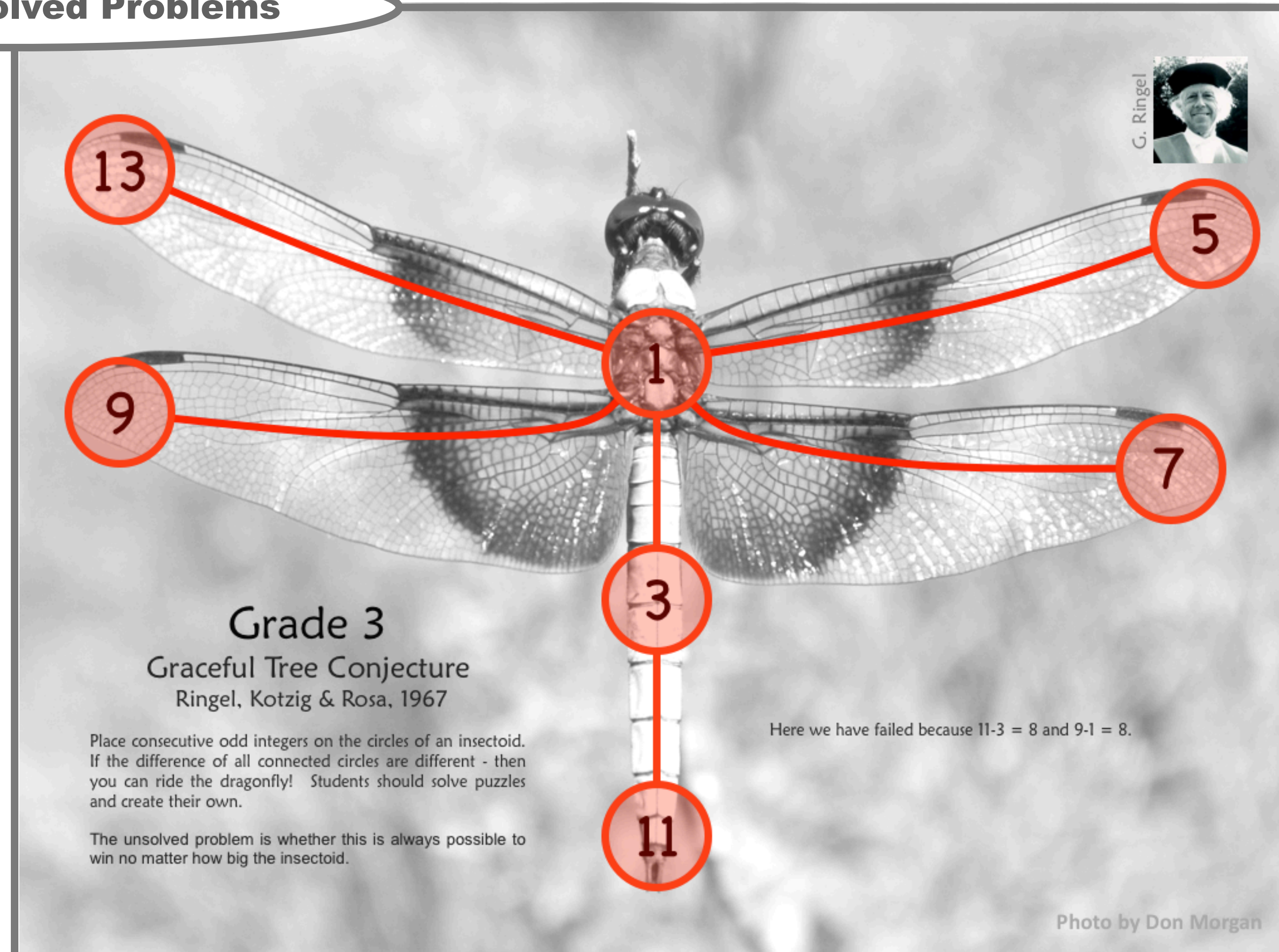
Grade 2

Sum-Free Partitions

Issai Schur, 1916

The witch adds frog #1 to a bubbling cauldron? Then frog #2. Then frog #3... How high can the witch go without getting gooped... without adding a number to a cauldron that has two numbers which add to it. For example, 24 cannot be added to the blue cauldron because 2+22 = 24. Grade 2 students start with two cauldrons and are asked to use eight frogs. Groups that are successful are quietly asked to get as high as possible with three cauldrons. An unsolved problem is to find the highest number that can be placed in 5 cauldrons.

G. Ringel



Grade 3

Graceful Tree Conjecture

Ringel, Kotzig & Rosa, 1967

Place consecutive odd integers on the circles of an insectoid. If the difference of all connected circles are different - then you can ride the dragonfly! Students should solve puzzles and create their own.

The unsolved problem is whether this is always possible to win no matter how big the insectoid.

Here we have failed because 11-3 = 8 and 9-1 = 8.

Photo by Don Morgan

Winning Unsolved Problems

Grade 4

Collatz Conjecture

Lothar Collatz, 1937

On the day before their fateful flight Icarus and Daedalus both have dreams. In Icarus' dream, he writes a number on a rock and hurls it off the tower where they have been imprisoned. If the number is even, it is halved. If it is odd, the number is tripled and one added to the result. For example if Icarus writes a 3 on the rock… you can follow the sequence of numbers until CRASH - he falls into the sea and is killed. This dream has turned into a nightmare. But Icarus knows that if he can just find a number to write on that rock so that he doesn't end up crashing into the sea… so that it doesn't end up at 1… then he will be all right. Daedalus has a similar dream. Your job is to help save the lives of Daedalus and Icarus.

Icarus

If even, then halve it.

If odd, then triple it and add one.

Daedalus

If even, then halve it.

If odd, then triple it and subtract one.

The unsolved problem is to find a way to save both lives. All grade 4 classes will discover a way to save Daedalus' life. How many numbers 20 and under have Daedalus surviving?

Grade 5 (grade 11 in US?)

Non-transitive dice

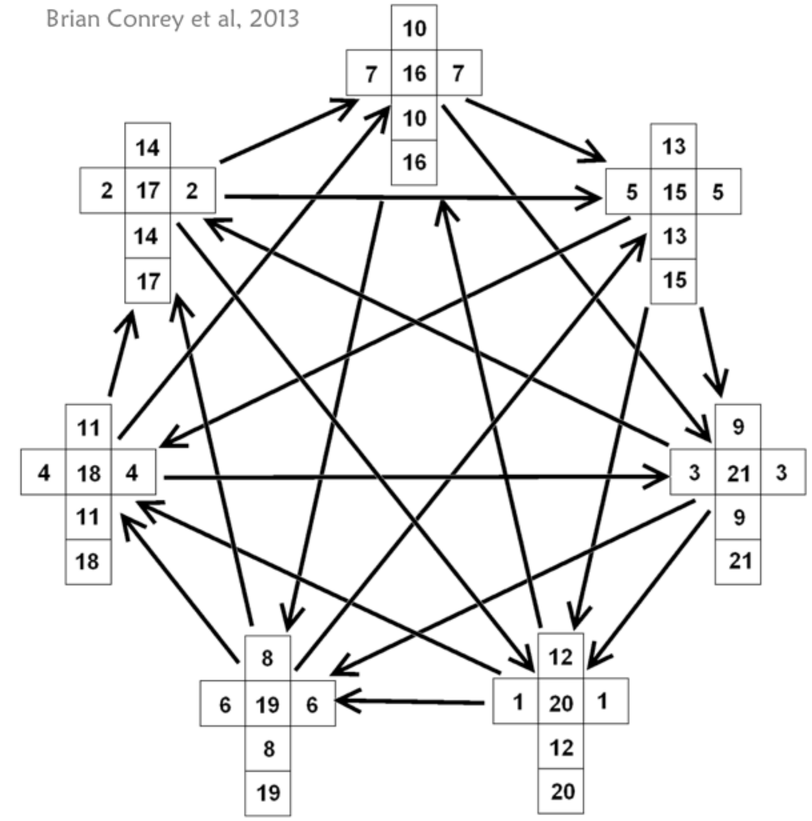
Brian Conrey et al, 2013

What fraction of 3 randomly generated n-sided dice with △n pips are non-transitive?

For six sided dice with 21 pips, this is close to 1/4. This fraction seems to be approached for large n. How intriguing!

Joshua Zucker recommended using 4 dice in the classroom just to ensure that the probability is reasonably high of an AHA! experience. In the elementary school classroom I give students a set of non-transitive dice and ask them to play against one-another. The loser after 7 rolls may opt to change dice with the winner. After 5 minutes I ask them for their opinion about which dice is best.

This problem can be adapted for the US market by asking what FRACTION of time Dice A beats Dice B. This is answered by the standard 6x6 probability grid. In many other countries probability is taught earlier than in the US.



Grade 6

Twin Prime Conjecture

Euclid, 250 BCE

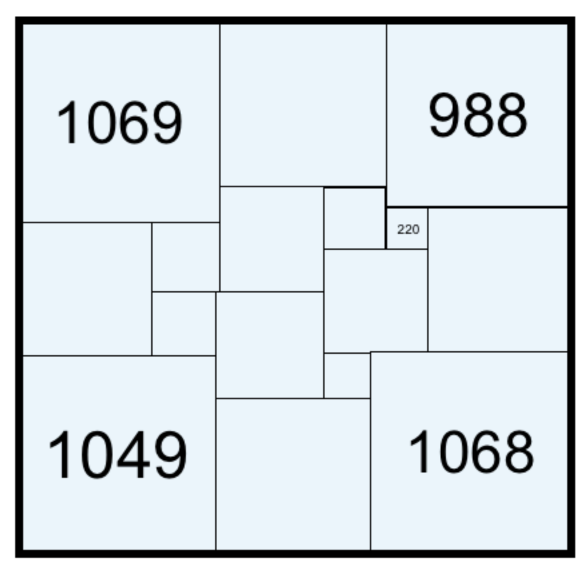
The twin prime conjecture states that there exist an infinite number of prime numbers, p, such that p+2 is prime. The lesson is based on Polya's paper: Heuristic Reasoning in the Theory of Numbers.

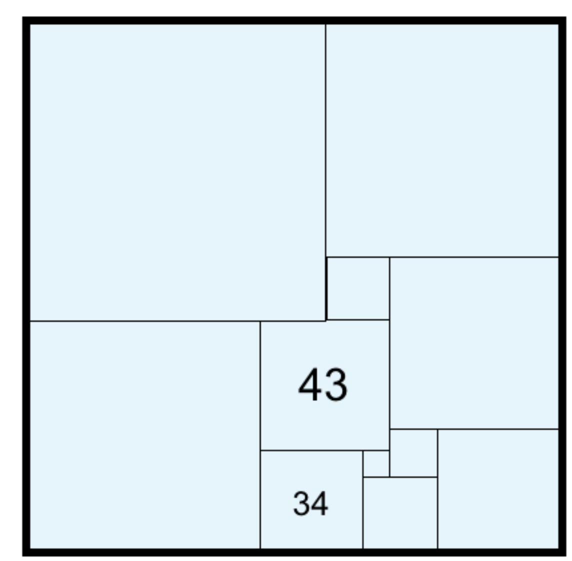
p	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53
p+2	5	7		13		19			31			43			
p+6		11	13	17	19	23		29		37	43	47		53	59

Grade 7

Tiling Rectangles with Squares of Different Sizes

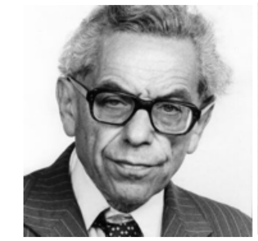
Stuart Anderson, 2013

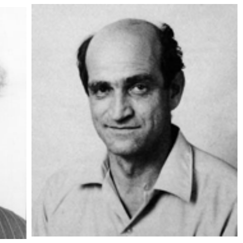




A number in a square represents its edge length. Give students a few numbers in squares and let them complete the rest. Give fewer clues to older students and they will need to use algebra. Younger students will be practicing addition and subtraction. The old unsolved problem from the 1930s was whether a square could be tiled using smaller squares of all different sizes (R. L. Brooks, C. A. B. Smith, A. H. Stone and W. T. Tutte). One new unsolved problem (not rigorously investigated) is what is the largest possible fraction: smallest square divided by largest square in any of these rectangles. The current record holder is the solution on the left which boasts an impressively large min/max = 220/1069.

http://www.squaring.net/sq/sr/spsr/spsr_minmax.html



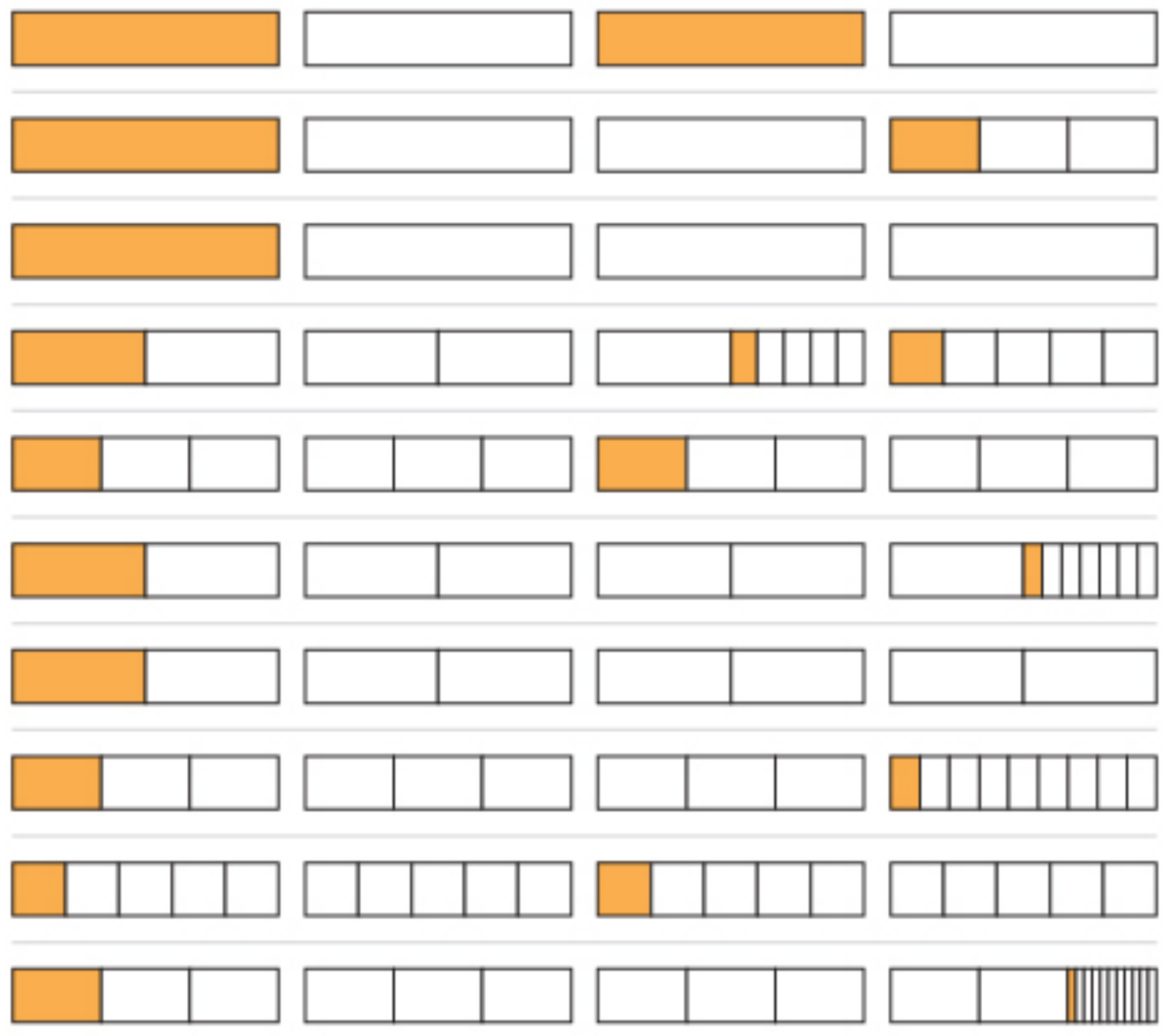


Grade 8

Erdős-Straus Conjecture

Paul Erdős & Ernst Straus, 1948

Andrzej Schinzel conjectures that for any positive k there exists a number N such that, for all $n \geq N$, there exists a solution in positive integers to $k/n = 1/x + 1/y + 1/z$. The version of this conjecture for $k = 4$ was made by Erdős and Straus and for $k = 5$ was made by Wacław Sierpiński.



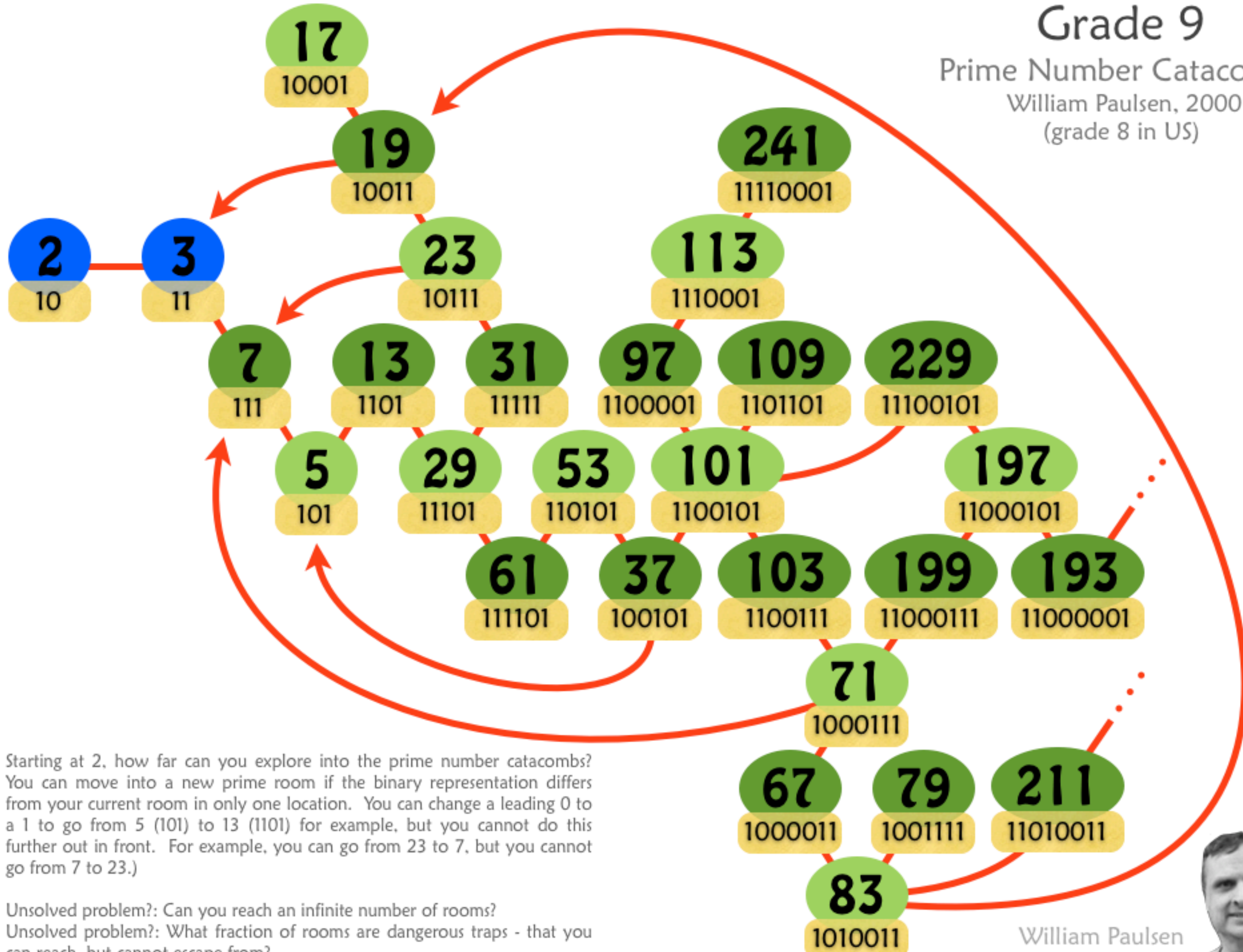
Puzzle by Joshua Zucker

Winning Unsolved Problems

Grade 9

Prime Number Catacombs

William Paulsen, 2000
(grade 8 in US)



Starting at 2, how far can you explore into the prime number catacombs? You can move into a new prime room if the binary representation differs from your current room in only one location. You can change a leading 0 to a 1 to go from 5 (101) to 13 (1101) for example, but you cannot do this further out in front. For example, you can go from 23 to 7, but you cannot go from 7 to 23.)

Unsolved problem?: Can you reach an infinite number of rooms?
Unsolved problem?: What fraction of rooms are dangerous traps - that you can reach, but cannot escape from?

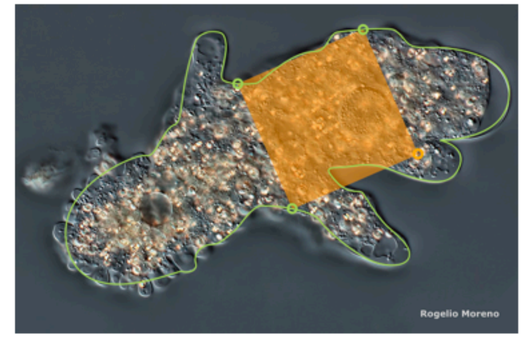
William Paulsen

Grade 10

Imbedded Square

Otto Toeplitz, 1911

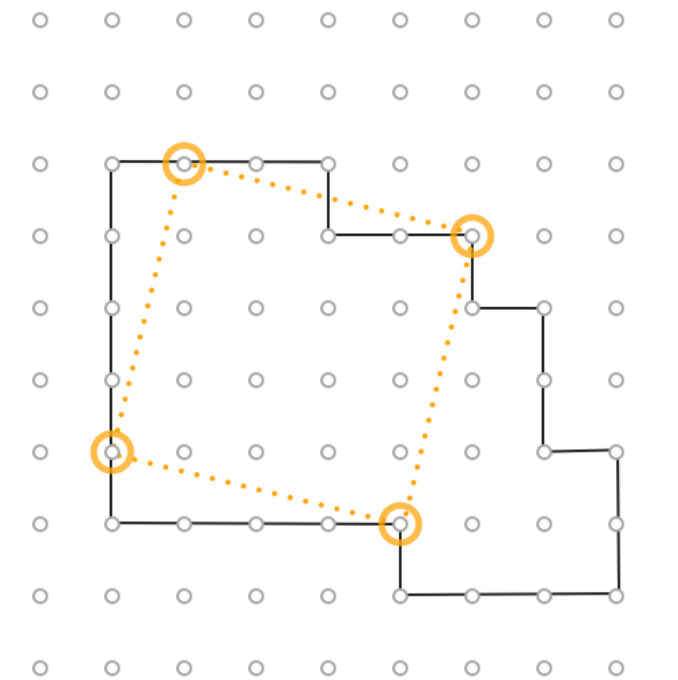
The class is told how to tickle their pet amoeba by placing a square blanket on it so that all four corners of the square blanket touch the perimeter of the amoeba. Is this always possible no matter what 2D shape the amoeba forms? This unsolved problem was posed in 1911.



The square blanket above fails because only three of its four corners tickle the perimeter of the amoeba.

A new problem: In taxi cab geometry the imbedded square problem becomes a curricular exercise to give students practice with perpendicular and parallel line segments in a Cartesian coordinate system. The challenge is for them to find some taxi cab loop which does not have an imbedded square (all four corners of the square on the intersections.) I could not find such a loop, but this problem has not been rigorously investigated.

Is there a taxi cab loop (other than the unit square) which does not support two imbedded squares? This has been solved - there do exist such loops and it is a fun challenge for students to find one of them.



Grade 11

Markov Numbers

Andrei Markov, 1879

There are an infinite number of Markov triples {x,y,z} with x≤y≤z that are solutions to the diophantine equation: $x^2 + y^2 + z^2 = 3xyz$. Is there a unique triple associated with each z? That's unsolved. This problem gives students great practice with the quadratic equation.

$$x^2 + y^2 + z^2 = 3xyz$$

{x,y,z} = {1,1,1}
↓
{1,1,2}
↓
{1,2,5}

↙ {1,5,13} ↘ {2,5,29}

{1,13,34} {5,13,194} {5,29,433} {2,29,169}

Grade 12

P=NP?

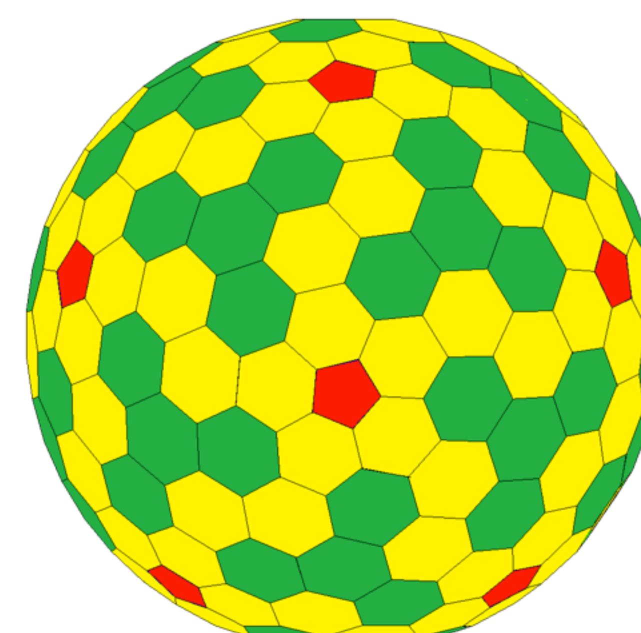
Stephen Cook, 1971

Let's duplicate this most important millennium problem. Grade 12 students are capable of understanding this even if this understanding is not curricular world wide.

Get students to try to solve certain classes of problems with varying n and then plot the time taken to solve versus n:

Finding if an integer, n, is odd or even.
Finding the number of digits of an integer n.
Finding the smallest element of a set.
Finding the number of vertices on a polygon.
Finding the number of vertices on a polyhedron.
Finding the median of a set of n numbers.
Adding two numbers with n digits.
Multiplying two numbers with n digits.
Determining if a graph of n vertices is bipartite.
Determining if a graph of n vertices is tripartite.
Finding the factors of a number n.
Finding a Hamiltonian cycle.

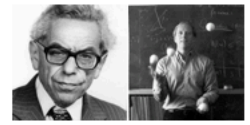
Students will get the idea of exponential versus polynomial times by painful experience! It is also nice for them to experience that while multiplying 5 digit numbers is more difficult than finding a hamiltonian cycle through 5 vertices, by the time you get to 100, the first is boring while the latter is potentially very, very difficult!



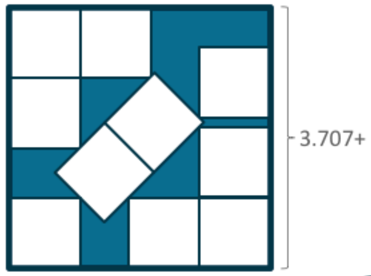
Kindergarten - Grade 1

Packing Squares

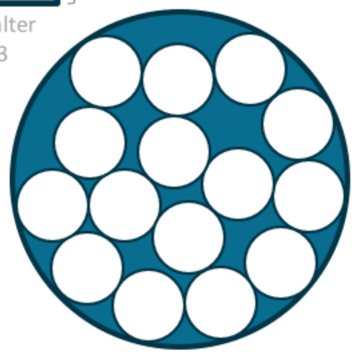
Erdős & Graham, 1975

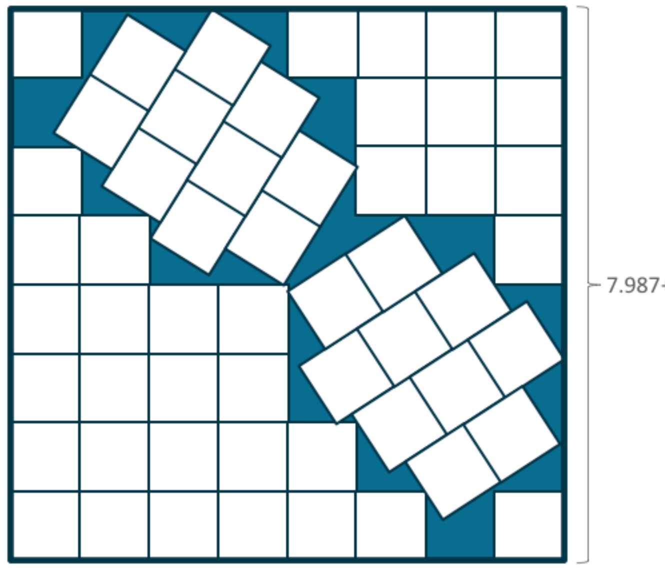


What is the smallest big square required to hold n unit squares? In general - how do you clean up and pack things away efficiently? The square in square problem has the advantage of being unsolved for n=11 unit squares and having a periodic structure - alternating between orthogonal and complex packing. Ed Pegg prefers the circle problem because coins are readily available to teachers.



n=10 proved by Walter Stromquist, 2003





n=55 found (but not proved) by David Cantrell, 2005

Erich Friedman's Packing Centre: www2.stetson.edu/~efriedma/packing.html

Willie Wiggle Wiggle Worm

Willie Wiggle Wiggle Worm got lost after a thunder shower one day so I made a little home for him out of a milk carton. He didn't know how to count, so I wrote the numbers 1 (head) to 25 (tail) on him. Each day I gave him a puzzle to help him practice counting and writing numbers.

The puzzles work like this: Some number hints are put in the carton. Willie Wiggle Wiggle Worm has to wind his way around in the carton so that the numbers on the carton are the same as the numbers on his body.

1				
10				7

Puzzle

1	2	3	4	5
22	23	24	25	6
21	10	9	8	7
20	11	12	13	14
19	18	17	16	15

Solution

Willie Wiggle Wiggle Worm does contort into fantastic shapes, but he can't go diagonally.

1	2	3	4	5
22	23	24	25	6
21	10	9	8	7
11	20	19	18	17
12	13	14	15	16

Wrong

7				
			17	
		11		
4				
1			24	25

★☆☆☆☆

by Lauren

		8			
		25			
		36			
		33			
		32			
		19			

★★☆☆☆

based on a puzzle by Isla, a kindergarten student

		20				
			30	31		
			1	42		
		11				

★★★☆☆

by Kindergarten students at River Valley School

		4		
			18	
		1		

★★★☆☆

by David, a kindergarten student

What are the minimum and maximum number of hints required to have a unique solution.

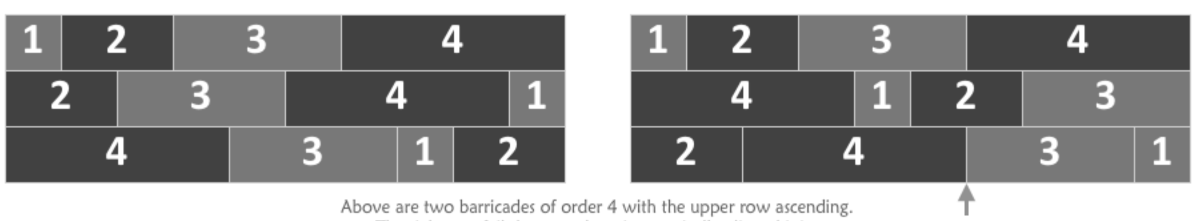
Other Unsolved Problems

Kindergarten - Grade 9

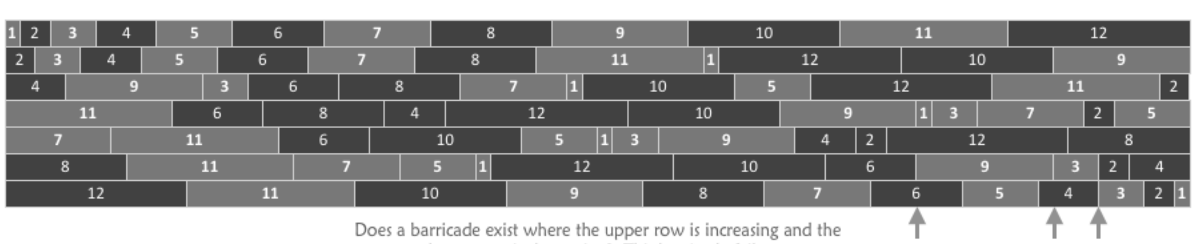
Building (Lego) Barricades

Barry Cipra, 2006

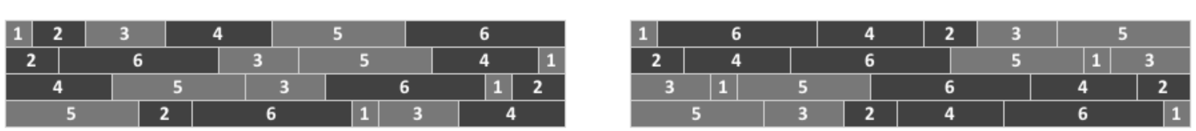
Barricades are constructed using logs of lengths 1 through n in each of (n+2)/2 rows. To make them strong enough to withstand projectiles, a barricade can not have two or more joins vertically aligned. There is some elegant algebra surrounding their construction which makes them a good choice for junior high. They are also great for grade 2s learning that addition is commutative and for younger students learning rules that apply within a system.



Above are two barricades of order 4 with the upper row ascending.
The right one fails because there is a vertically aligned join.



Does a barricade exist where the upper row is increasing and the lower row is decreasing? This barricade fails.



Above left is a barricade of order 6 with the upper row ascending. Above right is a barricade of order 6 that is special because it is rotationally symmetric.

Grade 1

Cookie Monster

Vanderlind, Guy, Larson, 2002

A cookie monster is presented with some jars with cookies in them. He wants to empty the jars in the fewest number of minutes possible. Each minute he may take the same number of cookies out from any number of the jars. What is the best strategy for him to empty a set of jars?

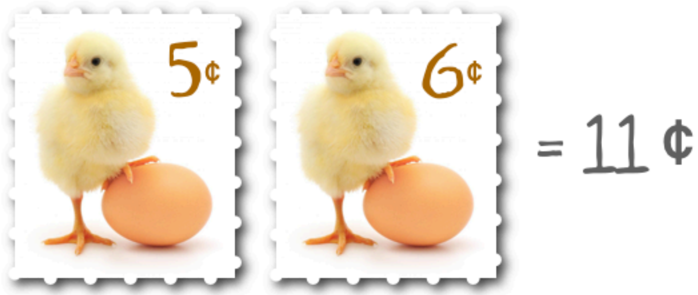
Grade 2

Postage stamp problem

Rohrbach, 1937

Find 6 denominations of stamps so that eggs can be mailed to the museum using at most two stamps. Start by trying to send a hummingbird egg (1₵) and work your way towards Ostrich eggs (50₵). What is the largest egg that you can send?

The unsolved problem is to find a general formula for the largest egg that can be posted as the number of denominations increases. The group of six denominations below can send eggs costing 1₵ through 12₵ - but not 13₵. This is far from optimal. Unsolved Problems in Number Theory (C-12) gives the solution: 1, 2, 5, 8, 9, 10 for six stamps and 1, 2, 3, 7, 11, 15, 19, 21, 22, 24 for 10 stamps.

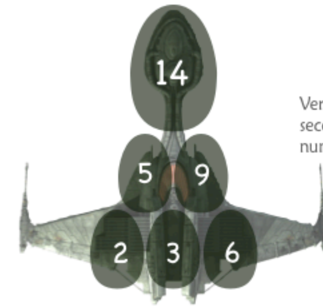


Grade 2

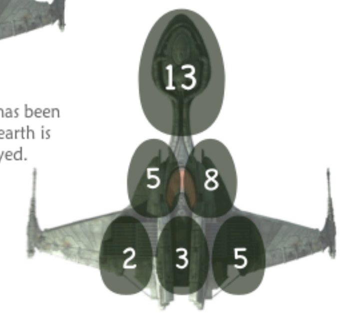
Klingon Attack

Spock, 1966

Very scary - Enemies are attacking earth. You can aim your ion cannon in one place each second. What is the fastest that you can destroy an enemy? A group of enemies? Each number must equal the sum of the two numbers underneath and must be unique.



Pseudo-Success! The spaceship has been destroyed in 14 seconds. The earth is saved, but Europe is destroyed.

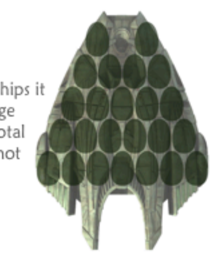


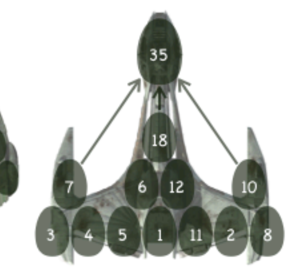
Failure! Duplicate 5s - the earth is destroyed.

Success! 8 seconds is the fastest time that this enemy can be destroyed.

Failure! Duplicate 14s. The earth is destroyed. These three bird-of-prey collectively have 18 targets and actually require 22 seconds to destroy. Try again!

The unsolved problem is which spaceships it is possible to destroy in sufficiently large numbers in n seconds where n is the total number of targets. This problem has not been rigorously investigated.



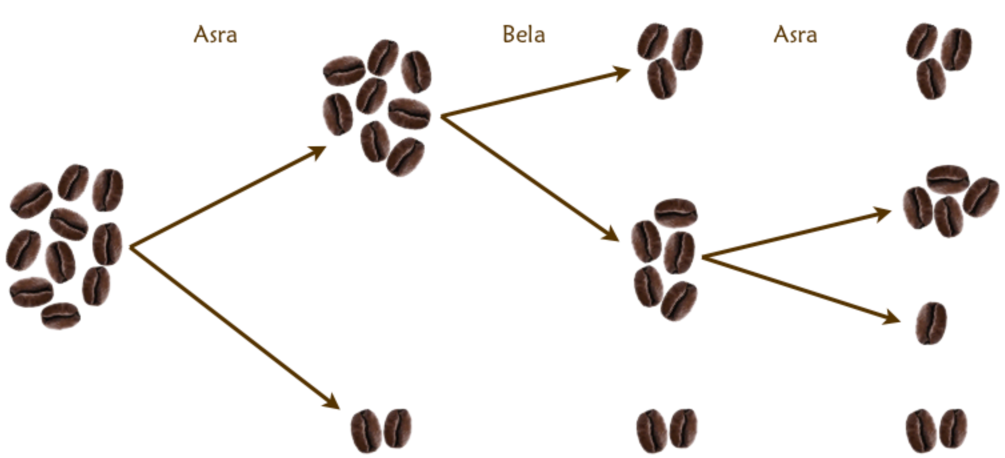


Grade 1

Squirel Nutkin Buries Nuts for Winter

Proposed by Richard Guy

Here is another possible addition to our already deep list of possibilities for grade 1. This is a two player game. Start with a heap of size n. On your turn you may split any heap into two parts... but you must be careful... after you've finished - no two heaps can be the same size. This is a GREAT game for grade 1. It is superior to all other nim games because the end-state is curricular. Children must compare quantities in different stacks. Apparently the game is also rich because there is a tantalizing link to triangular numbers (yet to be proved.) In the game below, Bela cannot go so Asra wins.



This game is not fun for most adults because it lacks strategy - it is only tactical - and the tactics become more and more mind-numbingly difficult as the number nuts / coffee beans increases. Contrast this to Aggression (grade 3) where players can have a strategic intuition that is accurate even for games on large maps. This criticism is not a negative in the grade 1 classroom.

Squirel Nutkin has not been rigorously investigated, so perhaps another unsolved representative from this group of NIM games should be selected - but the one above is definitely the game to present in the classroom. As a varient - consider the game played on a line. When you split a stack you move one part to the left along the line and one part to the right along the line. No two neighbouring piles may be the same size.

Grade 2

Magic Squares & Cubes

Ed Pegg wonders if there is a place for magic squares and cubes in our set of 13 unsolved problems. He writes: "For centuries, whether an order-five magic cube existed was unknown, until November 14, 2003, when C. Boyer and W. Trump discovered a solution. Note that the numbers 1 to 125 are used in this cube." To see a demonstration of this special cube, go to demonstrations.wolfram.com/MagicCubeOfOrder5

By deleting some entries in a = 4x4 magic square, puzzles can be created to help students practice addition and subtraction.

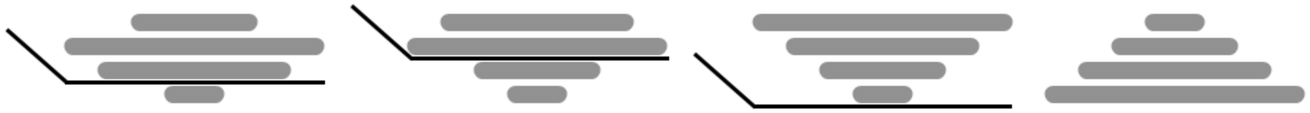
3	9	6	16
14	8	11	1
12	2	13	7
5	15	4	10

Grade 2

Pancake Flipping Problem

Gates & Papadimitriou, 1979

Find are the fewest number of flips to order pancakes into a beautiful pyramid.
For example, the left stack below takes three flips of the spatula.



Grade 2 students can challenge each other finding the three orderings of four pancakes which take 4 flips.

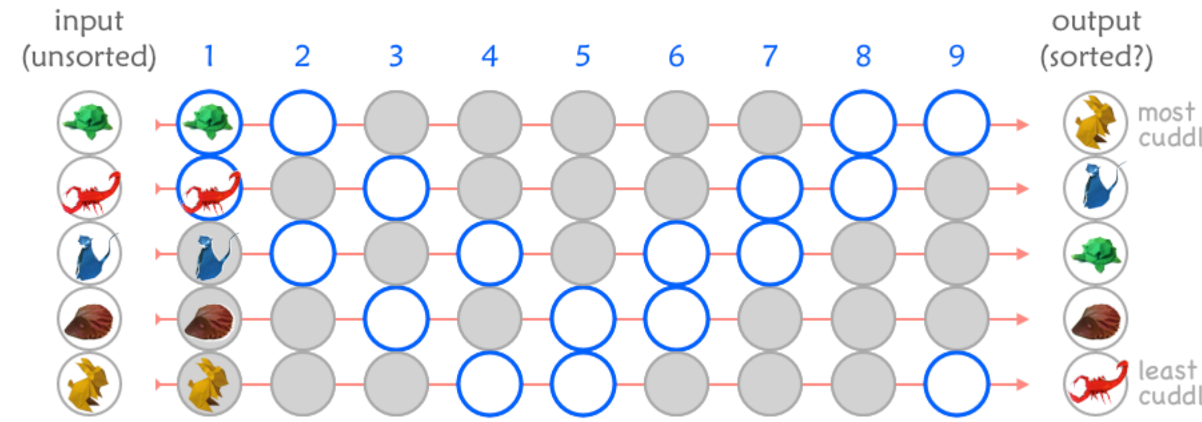
Another challenge is to ask the fewer number of flips to solve one of the stacks below. It is beyond the ability of most grade 2s to find the optimal solution (seven flips for both), but it is a good competition to establish the importance of creating a system to record results so that students can repeat a solution once discovered.

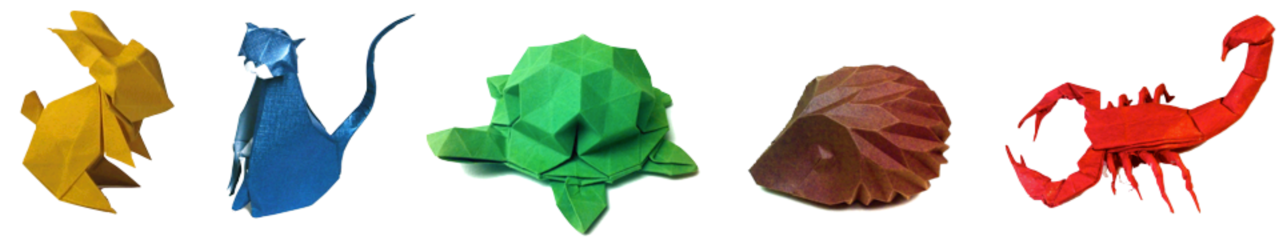
Grade 2

Sorting with fewest comparisons

Knuth, 1973

Lets say that you wanted to sort a bunch of origami animals on a cuddly scale from most cuddly to least cuddly. What are the fewest number of comparisons that need to be made to sort data? Students experiment with algorithms that work, but are too long, algorithms that don't work because they fail to sort all possible inputs, and then try to design their own optimal algorithm.





Grade 2

196 Palidromic Number Generator?

Gruenberger, 1984

Take any positive integer with two or more digits. Add it to the number obtained by reversing its digits. Continue until a palidromic number is obtained. Most small numbers terminate quite quickly. 89 is the first number that poses a real challenge… terminating with 88132000231188. Some numbers like 196 and 295 in base 10 may never become palindromic, but this has yet to be proved.

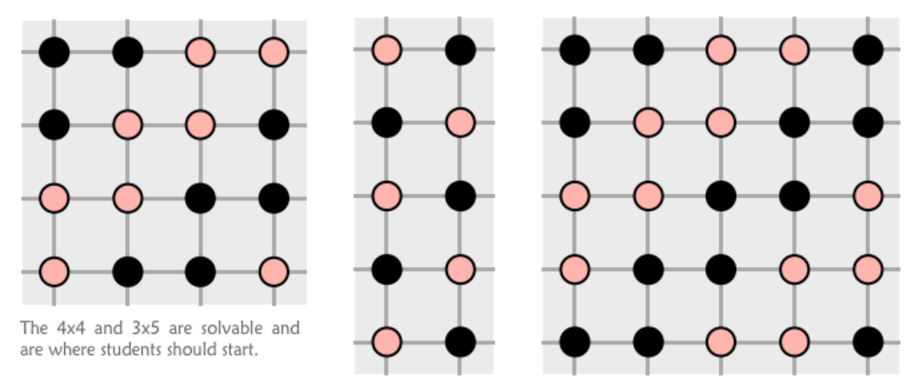
280	285	290	295	300	305	310	315	320
362	867	382	887	303	808	323	823	343
625	1635	665	1675					
1151	6996	1231	7436					
2662		2552	13783					
			52514					
			94039					
			187088					
			1067869					
			10755470					
			18211171					
			35322452					
			60744805					
			111589511					
			227574622					
			454050344					
			897100798					
			1794102596					

Grade 1-4

No Single-Colour Rectangle

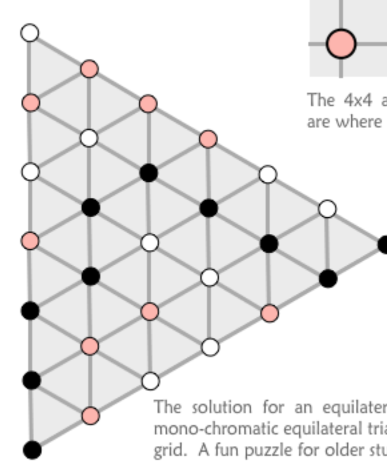
Bill Gasarch, 2009

Add a dark or light drop to each intersection. Can it be done so that no rectangle is created with all vertices the same colour? (Gasarch only considers rectangles with sides horizontal and vertical.)



The 4x4 and 3x5 are solvable and are where students should start.

An alternating pattern shows that any grid of size 2 x n is solvable. Grade 1 students are often capable of making this breakthrough. The 5x5 and 3x7 are both solvable, but too difficult for most grade 1 & 2 students. Here the 5x5 solution fails because there is a rectangle whose vertices are all the same colour.



The solution for an equilateral triangle of side-length 7 avoids all mono-chromatic equilateral triangles including those not parallel to the grid. A fun puzzle for older students, but too difficult for grade 1.

Gasarch has found most of the set of rectangles for which 4 colours are permitted. This set includes the 17x17 square. Partial solutions of these rectangles can be given - the students asked to fill in the remaining colours. The set of rectangles which can be solved with 5 colours is unknown.

Mutant Fibonacci Bunny Sequence

What is the fastest way to get to a target number (in this case 13) by repeatedly either adding the 1st three terms or finding the difference between the last two.

17

1 1 1 3 2 1 6 9 16 31 15 16 1 15 14 1 13

16

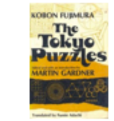
1 1 1 3 2 6 11 19 36 17 19 2 17 15 2 13

10

1 1 1 3 5 9 4 18 31 13

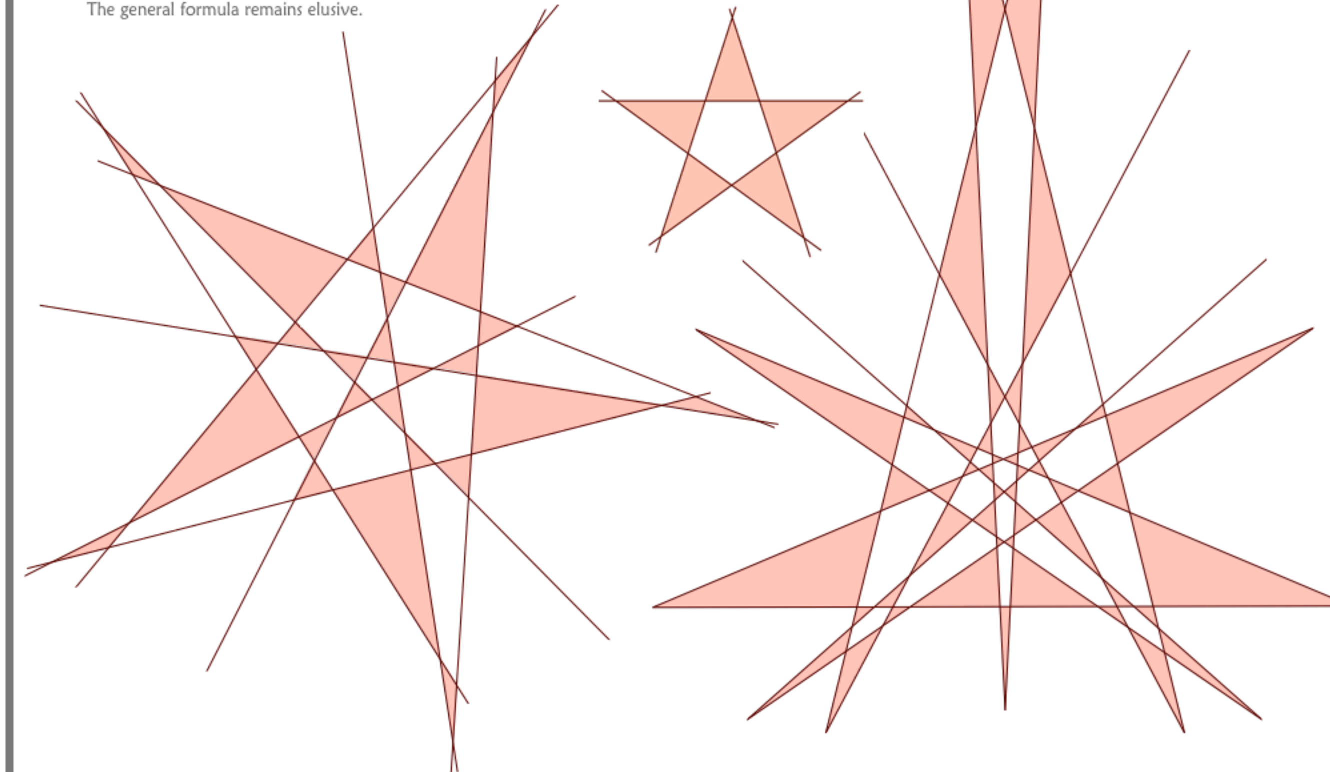
10

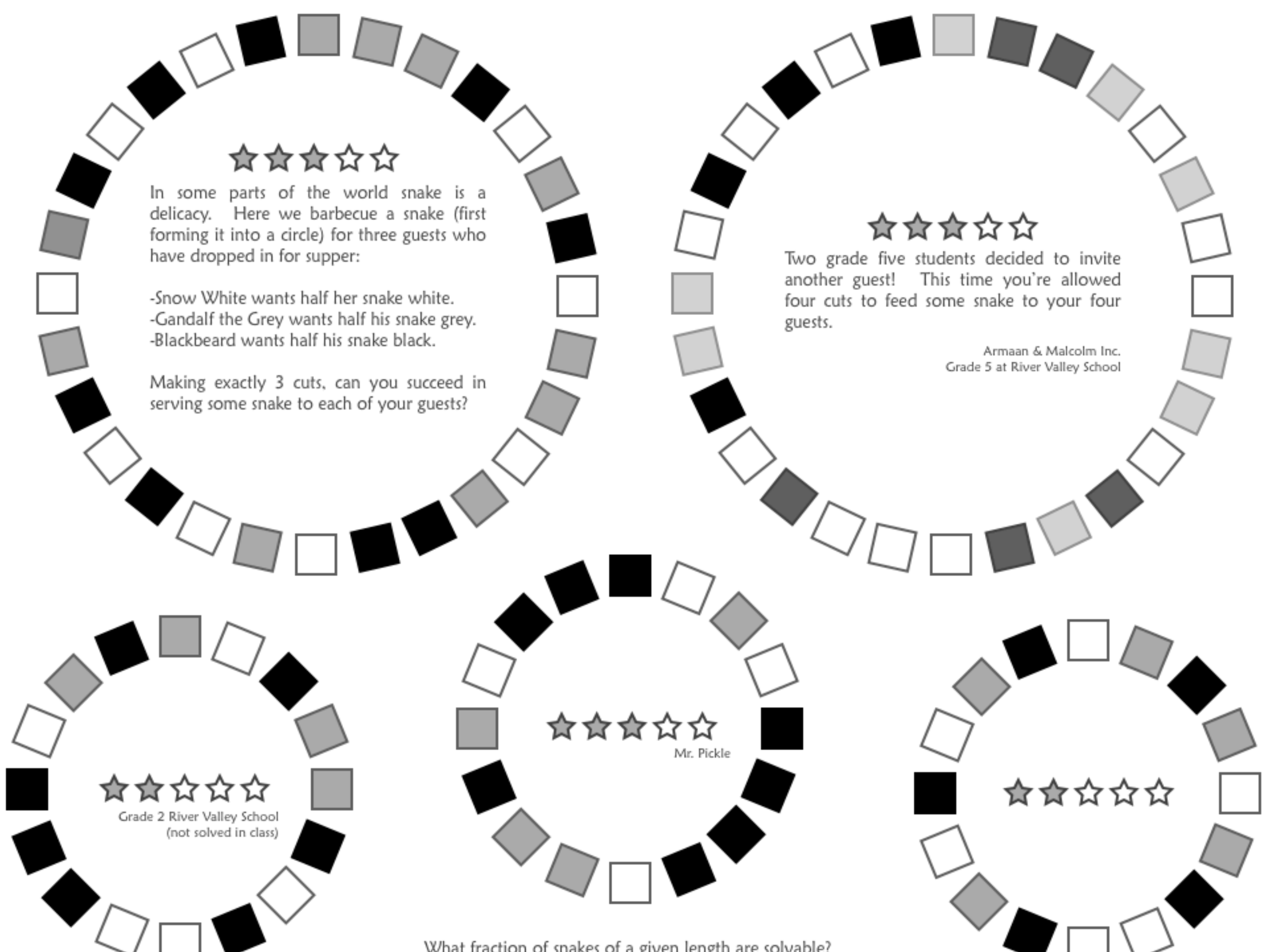
1 1 1 0 1 1 2 4 7 13



Grade 3
Kobon triangles
Kobon Fujimura, 1979

How many non-overlapping triangles can you create with n lines? Pictured are the currently best solution for 10 lines (25 triangles) below and a provably optimal solution for 5 lines (5 triangles) and 13 lines (47 triangles.) The general formula remains elusive.





★★★☆☆

In some parts of the world snake is a delicacy. Here we barbecue a snake (first forming it into a circle) for three guests who have dropped in for supper:

-Snow White wants half her snake white.
-Gandalf the Grey wants half his snake grey.
-Blackbeard wants half his snake black.

Making exactly 3 cuts, can you succeed in serving some snake to each of your guests?

★★★☆☆

Two grade five students decided to invite another guest! This time you're allowed four cuts to feed some snake to your four guests.

Armaan & Malcolm Inc.
Grade 5 at River Valley School

★★☆☆☆

Grade 2 River Valley School
(not solved in class)

★★★☆☆

Mr. Pickle

★★☆☆☆

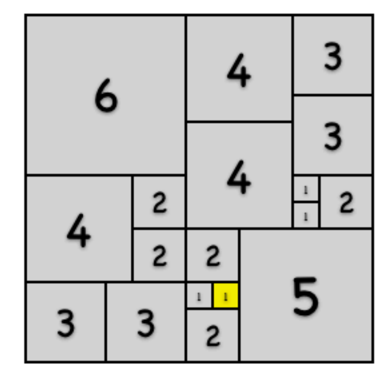
What fraction of snakes of a given length are solvable?

Grade 3
Mrs Perkins' Quilt
Henry Dudeney, 1917

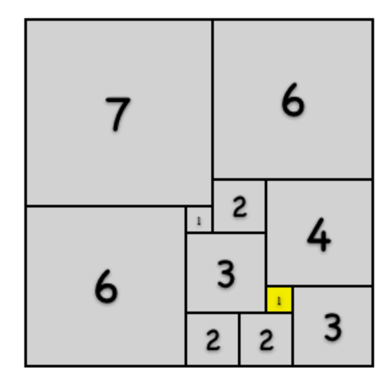
Ed Pegg Jr suggested the the group of square and rectangular problems related to Mrs. Perkins' Quilt. Here is how Henry Dudeney set it up for his readers: "For Christmas, Mrs. Potipher Perkins received a very pretty patchwork quilt constructed of 169 square pieces of silk material. The puzzle is to find the smallest number of square portions of which the quilt could be composed and show how they might be joined together. Or, to put it the reverse way, divide the quilt into as few square portions as possible by merely cutting the stitches."

This 13x13 problem has been generalized to squares of different sizes. The smaller squares need not be all different sizes, but they do need to be **relatively prime** to avoid trivial dissections like the dissection of a 4x4 square on the left.

This problem is great practice for grade 3 students adding and subtracting, but the idea of "relatively prime" is not curricularly appropriate. Therefore, the common generalization of Dudeney's puzzle should be amended so that a 1x1 gold tile MUST be included in every tiling of the teacher's new 5x5 bathroom / 7x7 bedroom / 9x9 kitchen / 11x11 garage / 13x13 living room. Young students understand this formulation and tiling is something more familiar to the bulk of students. Under this formulation the pictured 11x11 solution for the teacher's garage is no longer acceptable because we forgot the gold 1x1 tile.



18 tiles



11 tiles

Grade 4

Multiplicative Persistence

Gottlieb, 1969

Choose a positive integer. Multiply all the digits together. Repeat until you are left with a single digit. The number of steps that you've taken is equal to the multiplicative persistence of the number. What is the largest multiplicative persistence possible? Erdos asked questions when zeros were replaced by 1s… all numbers eventually crash to a single digit, but it can take a long time. If zeros are replaced by n, do numbers always crash. The answer is no for n = 15 for example

59 → 45 → 20 → 0
5 × 9, 4 × 5, 2 × 0

79 → 63 → 18 → 8
7 × 9, 6 × 3, 1 × 8

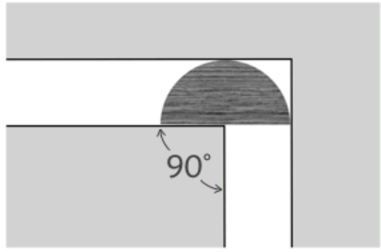
99 → 81 → 8
9 × 9, 8 × 1

Erdős asked what happens if zeros were always replaced by 1s… all numbers eventually crash to a single digit, but it can take time. If zeros are replaced by n, do numbers always crash? The answer is no for n = 15 (example: 4500 = 4*5*15*15), but seems to be yes for many larger n.

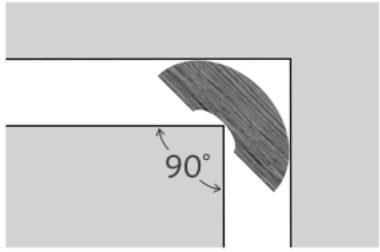
Grade 4,5

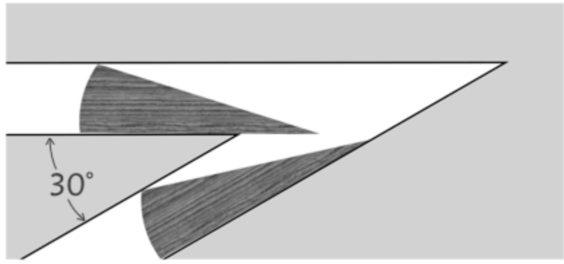
Moving Furniture

Leo Moser, 1966



During moving, you must drag various heavy desks around a ninety degree bend. What is the largest surface area of desk that you can accomplish this feat? Of course it can be done with the half circle desk shown on the left, but that is not optimal as can be seen with the larger desk on the right.





What is the angle for which the surface area of the maximal desk is minimized? At what angle does it become optimal to switch the leading edge to the lagging edge (see above)? These two questions have not been rigorously investigated.

Grade 4

n times Sum equals Product

Trost, 1956

Find a set of positive integers whose sum equals its product. For example, the three solutions for five elements in a set are: {2, 2, 2, 1, 1} which has a product and sum of 8, {3, 3, 1, 1, 1} which has a product and sum of 9, and {5, 2, 1, 1, 1} which gives 10.

Stating with n = 1, the number of different solutions with a set of n positive integers is: 1, 1, 1, 1, 3, 1, 2, 2, 2, 2, 3, 2, 4, 2, 2, 2, 4, 2, 4, 2, 4, 2, 4, 1, 5, 4, 3, 3, 5, 2, 4, 3, 5, 2, 3, 2, 6, 3, 3, 4, 7… (OEIS A033178) Ask students to discover patterns and prove that there is always at least one solution. This problem is D24 in RKG's "Unsolved Problems in Number Theory."

{2, 2, 2, 1, 1}

{3, 3, 1, 1, 1}

{5, 2, 1, 1, 1}

Grade 5

Sums Determining Members of a Set

Leo Moser, 1957

This problem is introduced by getting two children to secretly choose an integer each. They whisper to each other and then announce the result to the class. The class realizes that they cannot know for certain what the two numbers were. (Example: Fiona chooses -10 and Jane chooses +12. They announce "two" to the class who find it impossible to reconstruct the original numbers.)

Next three students are invited up and choose a number between about -5 and +5. Do not try a larger interval the first time playing unless you know your class ability well. All three group-whisper and announce the three sums resulting from adding up all possible pairings of their numbers. This time it is found that the three numbers can be calculated. The bulk of classroom time is spent exploring this three-person problem. (Example: Fiona chooses -2 and Jane and Bob both choose +3. They announce "one, one, six" to the class who can reconstruct the original numbers.)

Say "A group of four announce the following six pair-wise additions. What were their original numbers?" There are actually two possible solutions. The students must find both.

-2, -1, -1, +1, +1, +2

Show your students the following secret numbers that resulted from the game being played three times with eight people. The pair-wise additions are identical!

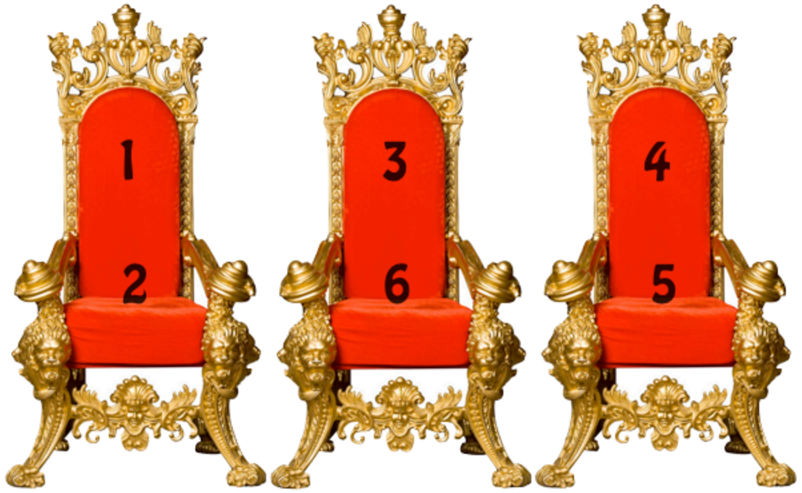
{±1, ±9, ±15, ±19} {±2, ±6, ±12, ±22} {±3, ±7, ±13, ±21}

Grade 7

Unfair Thrones

Royal Empress Menen I of Ethiopia, Empress of Empresses, Queen of Queens, 1903

Create n fractions by placing the integers {1, 2, 3... 2n} in either numerator or denominator. Your success comes by minimizing the difference of the greatest fraction minus the least. The puzzle is introduced by choosing a class empress and announcing that her royal highness has just given birth to twins. Two students are selected and come up to the front of the class to sit on their thrones. When the optimal solution for the twins is discovered… an announcement is made that the Empress has just given birth to a beautiful baby boy. Each new birth makes the problem more difficult.



The three - throne solution is shown above. The chance of civil war is 4/6 (the greatest) minus 1/2 (the least) = 1/6 or 17%.

The mathematician Charles Greathouse emailed me the following in June 2013 when I asked him if the problem is worthy of being selected:

"On one hand, it's in EXP with no obvious reduction ('might be hard'). On the other, I would not be surprised in the theory of Farey series could be brought to bear, so I wouldn't be shocked if it turned out to be easy…
On a quick analysis I find

0, 1/6, 1/6, 1/8, 6/35, 5/24, 3/14, 3/16, 1/6
as minimal scores for 1..2, 1..4, and so on."

Grade 7

Cupcake Problem

Alan Frank

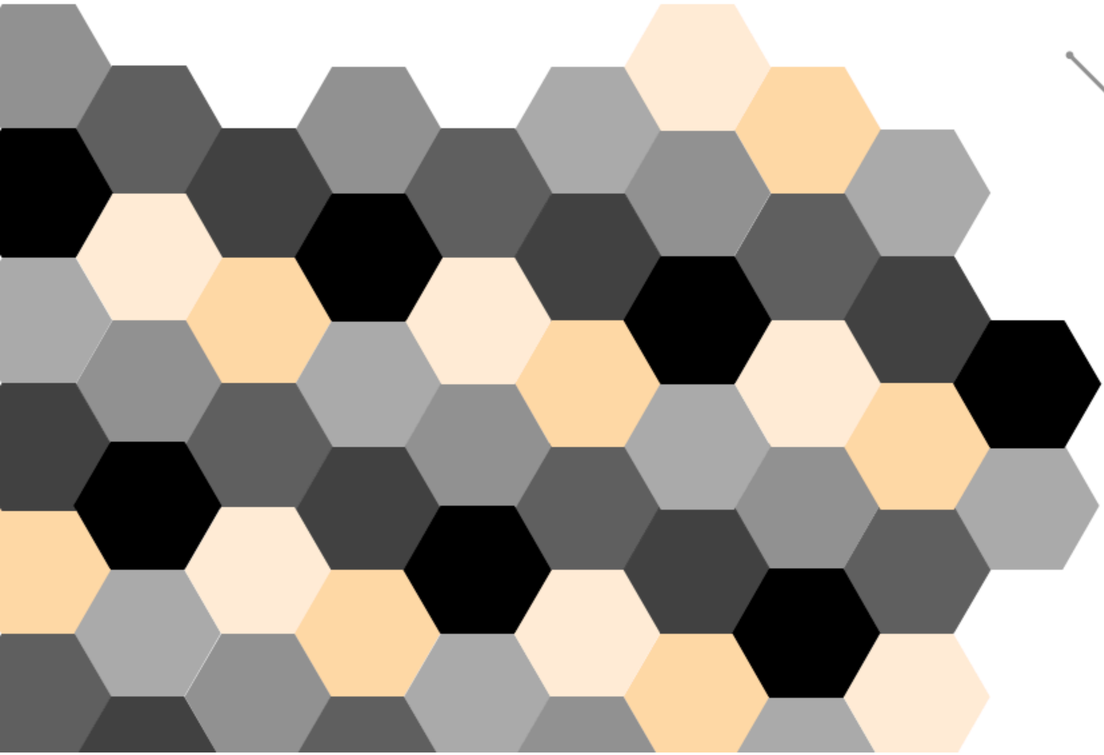
It's a birthday party gone awry. You've got 3 cupcakes but 5 children! Divide the cupcakes so that each child gets the same amount and the smallest piece is as large as possible.

Grade 7

Chromatic Number of the Plane

Hadwiger & Nelson, 1945

Place a toothpick on a student's page. If its ends are touching different colours, the student wins. One table of students beats another if they can win with fewer colours. This artistic problem also boasts some beautiful proofs to give an upper and lower bound.



Grade 8

Heilbronn Triangles in the Unit Square

Hans Heilbronn, c.1950

Ed Pegg Jr suggested the Heilbronn triangle problem: "For points in a square of side length one, find the three points that make the triangle with minimal area. Finding the placement of points that produces the largest such triangle is known as the Heilbronn triangle problem."

This problem could be developed either for grade 3 (concentrating on area), but it is going to be even more suitable for higher grades (concentrating on the formula for area and Pick's theorem). In either case a discrete version should be presented to the class… For example, to the left is a solution for the 7-point problem in a 5x5 square. How about 9 points in a 9x9 square below? Pick's theorem is marvelous in class - either as an accessible proof, or as an algebraic exercise where the theorem is given with missing variables. This unsolved problem may be just the excuse we need to promote Pick's theorem.

This problem also includes Dudeney's No-three-In a line problem.

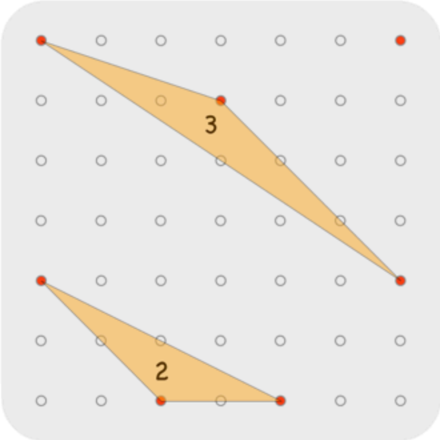
This problem is an ideal vehicle to introduce Pick's theorem. When I introduce Pick's theorem to students learning algebra, I first tell students that their may be a relationship of the form:

Area of quadrilaeral =
a*(# of lattice points on the perimeter) + b*(# of lattice points on the interior) + c.

… and ask them to find a, b and c if this is true. In fact this is true for simple (non-intersecting) polygons so my suggestion was incorrect and purposely misleading. Proving Pick's theorem is not too difficult and probably belongs in the K-12 mathematical experience.

Alternatively…

Area of polygon =
a*(# of lattice points on the perimeter) + b*(# of lattice points on the interior) + c*(# of verticies) + d.

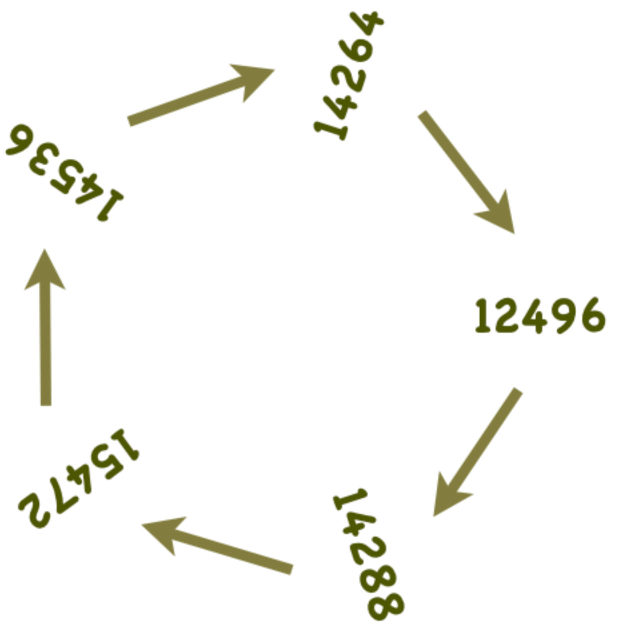


Grade 9

Perfect, Amicable and Sociable Numbers

Poulet, 1918

Perfect and amicable numbers have been known and celebrated since Classical times, but Sociable numbers were only discovered in 1918. Take a number n. Add its proper divisors. Repeat ad infinitum. If this iteration results in a cycle of period 1; you're sitting on a perfect number (6, 28, 496). If a cycle has period 2; you're sitting on amicable numbers ({220,284}, {1184, 1210}, {2620, 2924}). If a cycle has length more than 2; you're sitting on sociable numbers ({12496, 14288, 15472, 14536, 14264}, {14316, 19116, 31704, 47616, 833328, 177792, 295488, 629072, 589786, 294896, 358336, 418904, 366556, 274924,275444, 243760, 376736, 381028, 285778, 152990, 122410, 97946, 48976, 45946, 22976, 22744, 19916, 17716}). RKG Unsolved Problems in Number Theory presents many relevant unsolved problems.



Grade 9

Building with 1s

Richard Guy & John Conway, 1962

The class is split in two. Both halves choose numbers and see which half can reach them faster. One group can use 1s, brackets, addition and multiplication. The other half can use 1s, brackets, addition and powers, but not multiplication.

$$(1+1)(1+1+1+1+1+1+1)^{(1+1)}+1+1$$

$$((1+1+1)^{(1+1)}+1)^{(1+1)}$$

$$(1+1+1+1+1+1)^{(1+1)}+(1+1+1+1)^{(1+1+1)}$$

For example, a group of three students in the powers group tried to reach 100. The upper left one failed because it uses multiplication. The other two succeed, but the one on the bottom left is better because it uses only eight 1s. The multiplication half of the class cannot get to 100 in eight steps, so 100 is won by the powers group. 95 is won by the multiplication group. What is the largest number the class can find that is winnable by the multiplication group?

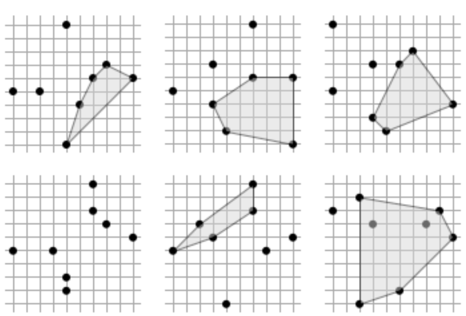
There are unsolved problems pertaining to the half of the class working with multiplication, but the use of using powers and not multiplication has not been rigorously investigated. Some related sequences in the OEIS are A003037, A005421, and A005520.

Grade 10

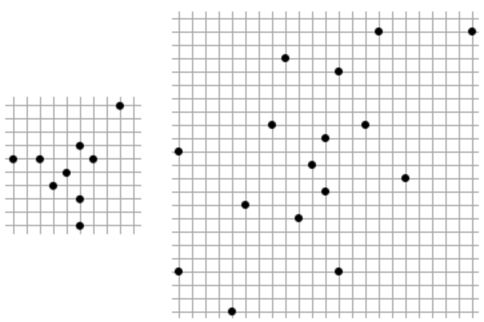
Convex Polygons

Erdös & Szekeres, 1935

What is the fewest number of points in general position (no 3 on a line) so that an n sided convex polygon is guaranteed by choosing a subset of the points as vertices. In 1935 Erdös & Szekeres conjectured that the answer was $2^{n-2}+1$ for $n\geq 3$. This conjecture still holds after Szekeres and Peters proved it correct for n=6 in 2006. It remains unsolved for all higher n.



Students experience this problem by attempting to find the number of points required for n=5. To add a stronger curricular link to Cartesian coordinates, the coordinates of each point must be integral and lie on one of the 100 intersections of a 10 x 10 grid.



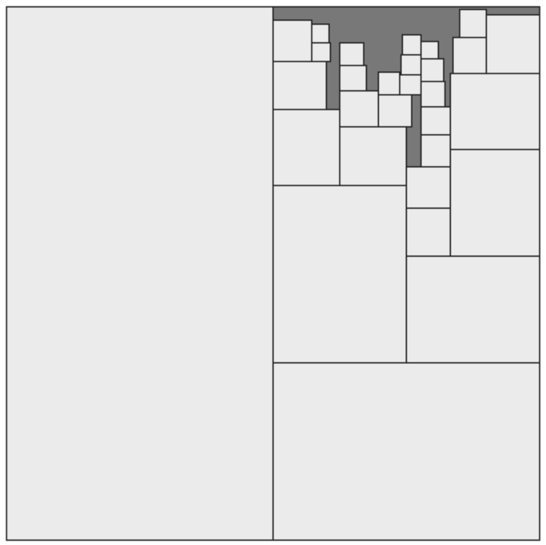
The little square on the left shows that 9 points are insufficient to guarantee that five of them form the vertices of an **empty** convex pentagon ("empty" means that none of the unused points are inside). The big square on the right show that 16 points are insufficient to guarantee that six among them form the vertices of an empty convex hexagon. It is not yet known if any number is sufficient to guarantee the existence of an empty hexagon.

Grade 10-12

Rectangling the Unit Square

It's beautiful to prove $\sum_{n=1}^{\infty} \frac{1}{n}\cdot\frac{1}{n+1} = 1$

Is it then possible to tile a unit square with an infinite number of rectangles of edge lengths 1/n and 1/(n+1)? If this is impossible - which is the first rectangle that cannot be squeezed in without overlapping a previous rectangle? This problem was suggested by Joshua Zucker.



Grade 6-12

Guilloché Patterns

Guillot, c.1620

Guilloché patterns are patterns created with imbedded cogs. A simplified two-cog version is available commercially under the name "spirograph". There is no known way to efficiently reconstruct the cogs that were used to create a Guilloché pattern - hence their traditional use on paper money to prevent forgery. This problem could be used with grade 6 students using spirograph and exploring the relationship between the number of teeth on the cogs and the number of revolutions required to complete a pattern. It could also be used with high school students studying polar coordinates. Ed Pegg Jr suggested this unsolved problem.